Classical damping torque. Analytical solution

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Classical damping torque as it is described in Landau–Lifshitz equation can be expressed as

$$\vec{T} = \lambda\left[\vec{M}\times\left[\vec{r}\times\vec{M}\right]\right] \quad (1)$$

where λ is the precession damping constant, which is assumed to be a constant; r is any vector in space. The torque T turns magnetization towards vector r independently of an initial direction of the magnetization M.

Note, for any realistic precession dumping both the precession damping constant λ and the magnetization M depend on the precession angle θ. However, in the classical calculation they both are assumed to be constants. This assumption is used in the calculateions below. .

First, let us rotate the coordinate system that the y-axis is along r vector. Then,

$$\vec{T} = \lambda\left[\vec{M}\times\left[\vec{y}\times\vec{M}\right]\right] \quad (2)$$

Next let us rotate coordinate system in the xz plane, so Mz=0
Then, the magnetization has x and y components as

$$\vec{M} = \begin{pmatrix} M_x \\ M_y \\ 0 \end{pmatrix}$$

$$\vec{y}\times\vec{M} = \begin{pmatrix} \vec{x} & \vec{y} & \vec{z} \\ 0 & 1 & 0 \\ M_x & M_y & 0 \end{pmatrix} = -M_x\cdot\vec{z}$$

The torque of Eq(2) is simplified as

$$\vec{T} = \lambda\left[\vec{M}\times\left[\vec{y}\times\vec{M}\right]\right] = \lambda\begin{pmatrix} \vec{x} & \vec{y} & \vec{z} \\ M_x & M_y & 0 \\ 0 & 0 & -M_x \end{pmatrix} = \lambda\left[-M_xM_y\cdot\vec{x} + M_x^2\cdot\vec{y}\right] \quad (3)$$

The dynamic equation for the magnetization precession damping can be written as

$$\frac{\partial\vec{M}}{\partial t} = \vec{T} = \lambda\left[-M_xM_y\cdot\vec{x} + M_x^2\cdot\vec{y}\right] \quad (4)$$

or explicitly

$$\begin{pmatrix} \dfrac{\partial M_x}{\partial t} \\ \dfrac{\partial M_y}{\partial t} \\ \dfrac{\partial M_z}{\partial t} \end{pmatrix} = \lambda\begin{pmatrix} -M_xM_y \\ M_x^2 \\ 0 \end{pmatrix} \quad (4a)$$

The solution of the 3rd equation of Eq.(4a) is Mz=0, since Mz=0 at t=0.
Instead Mx and My, new independences M and theta are introduced

$$\begin{aligned} M_x &= M\cdot\sin(\theta) \\ M_y &= M\cdot\cos(\theta) \end{aligned} \quad (6)$$

Then

$$\begin{aligned} \frac{\partial M_x}{\partial t} &= \sin(\theta)\frac{\partial M}{\partial t} + \cos(\theta)\frac{\partial\theta}{\partial t} \\ \frac{\partial M_y}{\partial t} &= \cos(\theta)\frac{\partial M}{\partial t} - \sin(\theta)\frac{\partial\theta}{\partial t} \end{aligned} \quad (7)$$

Substitution of Eq.(7) into 1st and 2nd Eqs. Of (4a) gives

$$\begin{aligned} \sin(\theta)\frac{\partial M}{\partial t} + \cos(\theta)\frac{\partial\theta}{\partial t} &= -\lambda\cdot M^2\sin(\theta)\cdot\cos(\theta) \\ \cos(\theta)\frac{\partial M}{\partial t} - \sin(\theta)\frac{\partial\theta}{\partial t} &= \lambda\cdot M^2\sin^2(\theta) \end{aligned} \quad (8)$$

Let us divide 1st Eq. on cos and 2nd Eq.on sin. Then

$$\begin{aligned} \frac{\sin(\theta)}{\cos(\theta)}\frac{\partial M}{\partial t}+M\frac{\partial \theta}{\partial t}&=-\lambda\cdot M^2\sin(\theta) \\ \frac{\cos(\theta)}{\sin(\theta)}\frac{\partial M}{\partial t}-M\frac{\partial \theta}{\partial t}&=\lambda\cdot M^2\sin(\theta) \end{aligned} \qquad (9)$$

Summing up two eqs of (9) gives

$$\left(\frac{\sin(\theta)}{\cos(\theta)}+\frac{\cos(\theta)}{\sin(\theta)}\right)\frac{\partial M}{\partial t}=0 \quad (10)$$

The solution of Eq.(10) is

$$\boxed{\frac{\partial M}{\partial t}=0 \quad (11)}$$

It means that the magnitude of the magnetization does not change in time as it is assumed in the classic mechanism of the precession damping.

Substitution of Eq.(11) into Eq.(9) gives

$$\begin{aligned} \cos(\theta)M\frac{\partial \theta}{\partial t}&=-\lambda\cdot M^2\sin(\theta)\cdot\cos(\theta) \\ -\sin(\theta)M\frac{\partial \theta}{\partial t}&=\lambda\cdot M^2\sin^2(\theta) \end{aligned} \qquad (12)$$

Two Eqs of (12) are exactly the same and can be written as

$$\frac{\partial \theta}{\partial t}=-\lambda M\cdot\sin(\theta) \quad (13)$$

Since

$$\int\frac{d\theta}{\sin(\theta)}=\log\left(\tan(\theta/2)\right) \quad (14)$$

The solution of Eq.(13) is

$$\log\left(\tan(\theta/2)\right)=-\lambda\cdot M\cdot t$$

or

$$\boxed{\theta=2\cdot\arctan\left(e^{-\lambda\cdot M\cdot t}\right)}$$