Analytical solution of the Landau-Lifshitz (LL) equations with conventional damping term

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The Landau-Lifshitz (LL) equations without the precession term can be written as

$$\frac{\partial \vec{m}}{\partial t} = -\lambda \cdot \vec{m} \times \left(\vec{m} \times \vec{H} \right) \quad (1)$$

where $\vec{m}$ is an unit vector directed along the magnetization $|\vec{m}| = 1$

Both the precession and the damping are induced by the same magnetic field H, which is directed along the z- axis.

$$\vec{H} = \begin{pmatrix} 0 \\ 0 \\ H_z \end{pmatrix}$$

Explicit expressions for vector products are

$$\vec{m} \times \vec{H} = \begin{pmatrix} m_y H_z \\ -m_x H_z \\ 0 \end{pmatrix} \quad \vec{m} \times \left(\vec{m} \times \vec{H} \right) = H_z \begin{pmatrix} m_x m_z \\ m_y m_z \\ -\left(m_x^2 + m_y^2 \right) \end{pmatrix} \quad (3)$$

Substitution Of Eq.(#) into Eq.(1) gives

$$\begin{aligned} \frac{\partial m_x}{\partial t} &= -\lambda H_z \cdot m_x m_z \\ \frac{\partial m_y}{\partial t} &= -\lambda H_z \cdot m_y m_z \qquad (4) \\ \frac{\partial m_z}{\partial t} &= \lambda H_z \cdot \left(m_x^2 + m_y^2 \right) \end{aligned}$$

Eqs.(4) can be divided into two systems, the first of which is independent of m_x and the second of which is independent of m_y. Taking m_y =0 in Eqs(4) gives

$$\frac{\partial m_x}{\partial t} = -\lambda H_z \cdot m_x m_z$$
$$\frac{\partial m_z}{\partial t} = \lambda H_z \cdot m_x^2 \qquad (5)$$

since the magnetization does not change its magnitude

$$|\vec{m}| = 1 \quad (6)$$

The solution of Eqs.(5) can be found as

$$m_z(t) = \cos(\theta(t))$$
$$m_x(t) = \sin(\theta(t)) \qquad (7)$$

where the θ is magnetization precession angle and is a new independent. Substitution of Eq.(7) into Eqs. (5) gives

$$\cos(\theta)\frac{\partial \theta}{\partial t} = -\lambda H_z \cdot \sin(\theta)\cos(\theta)$$
$$-\sin(\theta)\frac{\partial \theta}{\partial t} = \lambda H_z \cdot (\sin(\theta))^2 \qquad (8)$$

Two equations of (8) are identical and gives the damping torque as

$$\boxed{\frac{\partial \theta}{\partial t} = -\lambda(\theta) \cdot H_z \cdot \sin(\theta) \quad (9)}$$

The integration of Eq.(9) gives the temporal evolution of the precession angle as

$$\boxed{t = -\int_{\theta_0}^{\theta} \frac{d\theta}{\lambda(\theta) \cdot H_z \cdot \sin(\theta)} \quad (10)}$$

where θ_0 is the precession angle at time moment t=0.

Case 1: <u>the damping constant λ and H_z are independent of the precession angle</u>.

In case when both the λ and H_z are independent of θ, the damping torque can be calculated as

$$\boxed{\frac{\partial \theta}{\partial t} = -T_{\theta=90^0} \cdot \sin(\theta) \quad (20)}$$

where $T_{\theta=90}=\lambda H_z$ is the damping torque at θ=90 deg.
Taking into account that

$$\int \frac{d\theta}{\sin(\theta)} = \log\left(\tan\left(\frac{\theta}{2}\right)\right) \quad (21)$$

The temporal evolution is

$$\log\left(\tan\left(\frac{\theta}{2}\right)\right) - \log\left(\tan\left(\frac{\theta_0}{2}\right)\right) = -\lambda H_z t \quad (22)$$

or

$$\boxed{\tan\left(\frac{\theta}{2}\right) = e^{-\lambda H_z t}\tan\left(\frac{\theta_0}{2}\right) \quad (23)}$$

Case 2 λ is independent of the precession angle. Internal field depends on the magnetization angle.

The total magnetic field H consists of the external magnetic field and the internal magnetic field, which is proportional to the z-component of the magnetization. When the precession angle increase and the magnetization is tilted from z-direction, the z-component of the magnetization and correspondently the z-component of the internal magnetic field decreases.

In this case, the z- component H_z of the total magnetic field depends on the precession angle as:

$$H_z = H_{ext,z} + H_M \cdot \cos(\theta) \quad (30)$$

where $H_{ext,z}$ is the external magnetic field, H_M is the internal magnetic field, which is proportional to the z-component of the magnetization.