Spinor & Spin Direction

The spinor is an eigen vector of the Pauli matrix:

There are three Pauli matrixes corresponded to each direction:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The spinor is an eigenvector of a Pauli matrix. A quantum state of an electron, which spin direction is at angles θ and ϕ as shown in Fig.5, is described by a spinor S, which is eigen vector of the following Pauli matrix:

$$\sigma = \left[\sigma_x \cos(\phi) + \sigma_y \sin(\phi)\right]\sin(\theta) + \sigma_z \cos(\theta) =$$

$$= \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\cos(\phi) + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\sin(\phi)\right]\sin(\theta) + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cos(\theta)$$

Eigenvalue of any Pauli matrix is 1. Therefore, the spinor (eigenvector) $\begin{pmatrix} a \\ b \end{pmatrix}$ can be found from equation

$$\sigma \cdot S = S$$

To make the calculation simple, at first the spinor is found for a simplified case

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Simplified case 1.

The spin direction is the xy plane. θ= π/2

The Pauli matrix is

$$\sigma_{xy} = \sigma_x \cos(\phi) + \sigma_y \sin(\phi) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\cos(\phi) + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\sin(\phi) = \begin{pmatrix} 0 & e^{-i\cdot\phi} \\ e^{i\cdot\phi} & 0 \end{pmatrix}$$

The spinor S_{xy} is eigenvector of σ_{xy}

$$\sigma_{xy} \cdot S_{xy} = S_{xy}$$

$$S_{xy} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ e^{i\phi} \end{pmatrix}$$

When spin is along the x axis ϕ=0

$$S_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

When spin is opposite to the x axis ϕ=π

$$S_{-x} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

When spin is along the y- axis $\phi=\pi/2$

$$S_{y} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

When spin is along the y- axis $\phi=\pi/2$

$$S_{-y} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

- <u>Simplified case 2</u>. The spin direction is the xz plane. $\phi=0$

$$\sigma_{xz} = \sigma_x \sin(\theta) + \sigma_z \cos(\theta) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sin(\theta) + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cos(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix}$$

The spinor (eigenvector) of is

$$S_{xz} = \frac{1}{\sqrt{2(1-\cos(\theta))}}\begin{pmatrix} \sin(\theta) \\ 1-\cos(\theta) \end{pmatrix} \quad \theta \neq 0$$

or

$$S_{xz} = \frac{1}{\sqrt{2(1+\cos(\theta))}}\begin{pmatrix} 1+\cos(\theta) \\ \sin(\theta) \end{pmatrix} \quad \theta \neq \pi$$

When spin is along the z- axis $\theta=0$

$$S_{z} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

When spin is opposite to the z- axis $\theta=\pi$

$$S_{z} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

When spin is along the x axis $\theta=\pi/2$

$$S_{x} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

When spin is opposite to the x axis $\theta=-\pi/2$

$$S_{-x} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

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general case

$$S = \frac{1}{\sqrt{2\left(1-\cos(\theta)\right)}} \begin{pmatrix} \sin(\theta) \\ \left(1-\cos(\theta)\right)e^{i\phi} \end{pmatrix} \quad \theta \neq 0$$

or

$$S = \frac{1}{\sqrt{2\left(1+\cos(\theta)\right)}} \begin{pmatrix} 1+\cos(\theta) \\ \sin(\theta)e^{i\phi} \end{pmatrix} \quad \theta \neq \pi$$